accurate than the linear approximation proposed in Ref. 1. However, a word of caution is necessary. It was shown in Ref. 7 (and also in Ref. 2) that the quality of extrapolations based on first-order sensitivity derivatives may be reduced greatly when parameter variations cause changes in the membership of the set of the active constraints. If such changes occur, the range of accurate extrapolation may not be extended by use of quadratic extrapolation based on $d^2\bar{F}/dp^2$.

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Vorticity at the Shock Foot in Inviscid Flow

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Introduction

It is characteristic of transonic flows to have a shock or shocks embedded in the flowfield. The flow immediately behind the shock is related to the flow ahead of it by the Rankine-Hugoniot conditions and is a function of the shock shape. If the shock is normal to the body surface, the flow behind the shock will be subsonic and its shape will, in general, not be known a priori. The shock shape must be determined in conjunction with the local flowfield.

Although inviscid transonic flow past a body can be computed routinely using numerical methods developed in the last decade, the shock region has always been the most erroneous part of the solution. Often the potential approximation is made, and this is inconsistent with the conservation of normal momentum across the shock. Also, only the weak form of the governing equation is satisfied to an order normally so restricted by numerical stability that the actual shock is smeared over several grid points. As the shock gets stronger, entropy and vorticity production behind the shock can no longer be ignored and potential theory fails. These effects, added to the already complex flowfield, makes the construction of a proper numerical scheme a difficult task.

In this Note we study the flowfield immediately downstream of a shock at its root where the shock meets a smooth convex surface. Lin and Rubinov[1] first noted that a singularity occurs at the shock root. They also argued that the shock shape at the root must be of the form

$$\xi = k\eta^{3/2}$$

where ξ, η are coordinates of the shock measured along and normal to the body, respectively. Zierep[2] also found the same shock shape but was unable to determine the constant k for a convex body. Later Gadd[3] pointed out that the flow behind and at the shock root, determined by the Rankine-Hugoniot conditions, experiences a discontinuity in curvature in order to conform to the body. Such a flow is known to have a multivalued normal pressure gradient and a streamwise pressure gradient that is logarithmically singular.

We shall determine the vorticity behind the curved shock at this singular point and discuss to what extent the flowfield is affected by this vorticity.

Shock-Induced Vorticity

It is well known that the vorticity behind a shock can be computed by applying Crocco's theorem, i.e.,

$$\zeta_2 = -\frac{1}{q_2\sin(\sigma-\alpha)}\left[\frac{1}{2}\frac{dq_2^2}{d\ell} + \frac{1}{\rho_2}\frac{dp_2}{d\ell}\right] \tag{1}$$

where subscript 2 denotes quantities evaluated after the shock, ζ is the vorticity induced by the shock, q the flow speed, ρ the density, p the pressure, ℓ the distance along the shock, σ the shock angle measured relative to the upstream flow, and α the flow deflection angle after the shock (Fig. 1).

The Rankine-Hugoniot conditions require that the post-shock quantities be related to preshock quantities as follows:

$$q_2^2 = q_1^2[1-(1-\epsilon^2)\sin^2\sigma]$$

$$p_2 = p_1 + \rho_1 q_1^2 (1-\epsilon)\sin^2\sigma$$

$$\epsilon = \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2\sin^2\sigma} \tag{2}$$

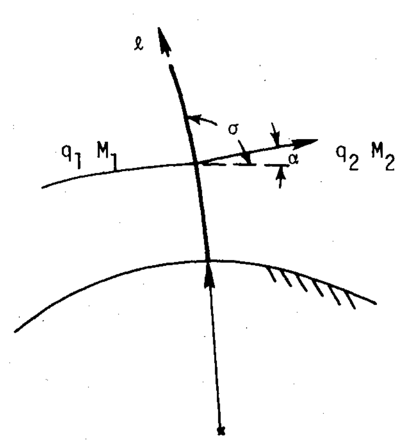

Fig. 1 Flow over a body with a shock.

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Dedicated to Professor William R. Sears in celebration of his 70th birthday, with the author's admiration.

where the subscript 1 denotes quantities ahead of the shock, γ is the ratio of the specific heats, and M is the Mach number. After substituting Eq. (2) into Eq. (1), we see that the vorticity equation becomes

$$\zeta_2 = -\frac{(1-\epsilon)q_1}{\epsilon\sin\sigma}\left\{[\epsilon(M_1^2-1)\sin^2\sigma - \cos\sigma]\frac{1}{\rho_1 q_1^2}\frac{dp_1}{d\ell} - (1-\epsilon)\sin\sigma\cos\sigma\frac{d\sigma}{d\ell}\right\} \tag{3}$$

At the shock root where the shock is normal this formula can be further simplified to

$$\zeta_2 = -\frac{(1-\epsilon)q_1}{\epsilon}\left[\frac{\epsilon(M_1^2-1)}{R} - (1-\epsilon)\cos\sigma\frac{d\sigma}{d\ell}\right] \tag{4}$$

where R is the radius of curvature of the body at the shock root. We note that the last term on the right-hand side of Eq. (4) cannot be evaluated immediately but $\cos\sigma(d\sigma/d\ell)$ must be negative since $\sin\sigma$ attains its maximum at the root with $\sigma=\pi/2$. We thus conclude that the vorticity is clockwise or negative, and is a second-order quantity proportional to $(1-\epsilon)^2$ since, from Eq. (2),

$$1-\epsilon = \frac{2(M_1^2\sin^2\sigma - 1)}{(\gamma+1)M_1^2\sin^2\sigma} \tag{5}$$

In order to evaluate the vorticity, the shock shape $\sigma(\ell)$ must be found. We note here that the potential approximation is only valid to second order at the shock root unless $1/R = 0(1-\epsilon)$, i.e., unless the slender body approximation is made.

Flow Behind the Shock

The inviscid subsonic flow after the shock would be rotational in general and therefore is governed by the Euler equations. The continuity equation, written in intrinsic coordinates s and n, along and normal to flow, is

$$\frac{\partial \rho q}{\partial s} + \rho q\frac{\partial\theta}{\partial n} = 0 \tag{6}$$

The vorticity equation by definition is

$$\zeta = q\frac{\partial\theta}{\partial s} - \frac{\partial q}{\partial n} \tag{7}$$

where θ is the flow angle measured with a fixed reference frame.

If one defines an intrinsic stream function Ψ as

$$\Psi_n = \ell n\rho q \quad\text{and}\quad \Psi_s = -\theta$$

then, Eq. (6) is automatically satisfied and Eq. (7) becomes

$$(1-M^2)\Psi_{ss} + \Psi_{nn} = -\frac{[1+(\gamma-1)M^2]}{q}\zeta \tag{8}$$

This equation can be solved for Ψ with given ζ as in Eq. (4) by specifying Ψ at $n=0$; Ψ_s and Ψ_n at the shock boundary $\ell(s,n)$ with Ψ_s and Ψ_n being finite at downstream infinity. The requirement that this elliptic system is not being over-specified thus determines the shock.

In general, Eq. (8) can only be solved by numerical means; however, a local expansion, making use of the fact that the vorticity is only a second-order quantity as mentioned earlier, can be very informative in understanding the flow behind a shock.

Vorticity at the Shock Root

Since vorticity is shown to be of the order of $(1-\epsilon)^2$, a formal expansion, e.g., $\Psi = \Psi^0 + \delta\Psi^1 + \ldots$ in a small parameter $\delta = \delta(1-\epsilon)$, yields the lowest order equation for Ψ valid at the vicinity of the shock root as follows

$$(1-M_2^2)\Psi_{ss}^0 + \Psi_{nn}^0 = 0 \tag{9}$$

The solution of this equation subject to the stated boundary conditions was first given by Gadd[2] and later solved in a more formal manner by Oswatitsch and Zierep.[3] The shock shape they found, which is different from that predicted by Lin and Rubinov,[1] is normal to the body and has a logarithmic singularity in curvature at the body. The unknown quantity, $\cos\sigma(d\sigma/d\ell)$, evaluated at the shock root is then zero.

We conclude that since vorticity is second order the flowfield behind the shock remains irrotational to the lowest order. The local curvature of shock at the shock root, despite being logarithmical-infinite, does not contribute to the vorticity. For slender bodies the shock curvature as well as the body curvature are $\mathcal{O}(1-\epsilon)$. Thus our result for vorticity reduces to the usual small disturbance result of $(1-\epsilon)^3$.

Since all the higher order equations are of Poisson-type with homogeneous boundary conditions, their solutions should be regular and have no more singular contribution to the shock curvature at the root than the lowest order logarithmic one. We then obtain the equation for vorticity at the root as

$$\zeta_2 = -\frac{(1-\epsilon)q_1}{R}(M_1^2-1) \tag{10}$$

where

$$\epsilon = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2}$$

in terms of known quantities q_1 and M_1, upstream of the shock and the body curvature $1/R$.

Conclusion

We have shown that vorticity behind the shock is of the order of $(1-\epsilon)^2$ even at the shock root where the curvature of the shock is infinite, and have obtained a formula for this vorticity valid at the root in terms of upstream quantities only. This formula can be used in the construction of an accurate numerical scheme for Eq. (8).

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